

# SAMPLE COVARIANCE MATRICES CONVERGE TO COMPOUND FREE POISSON DISTRIBUTION

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**ABSTRACT.** We show that the empirical distribution of the eigenvalues of the sample covariance matrix of certain random vectors (not necessarily independent entries) with bounded marginal  $L^4$  norms converges weakly to a compound free Poisson distribution.

## 1. MAIN RESULT

Marchenko and Pastur [2] showed that the empirical distribution of the eigenvalues of the sample covariance matrix of a random vector uniformly distributed on the unit sphere converges weakly to the Marchenko-Pastur law. There has been many generalizations to general random vectors (see [1]). The main result of this paper is

**Theorem 1.1.** *Suppose that  $f_1, \dots, f_N$  are independent random vectors on  $\mathbb{C}^n$  such that*

$$\sup_{x \in S^{n-1}} \mathbb{E}|(f_j, x)|^4 \leq \frac{L}{n^2} \text{ and } \mathbb{E}\|f_j\|^k \leq L_k, \quad j = 1, \dots, N, \quad k \geq 1$$

*for some  $L > 0$  and  $L_k > 0$ ,  $k \geq 1$  independent of  $n$  and  $N$ . If  $n, N \rightarrow \infty$  in such a way that  $\frac{n}{N} \rightarrow \lambda \in (0, \infty)$  and*

$$\left\| \sum_{j=1}^N \mathbb{E}\|f_j\|^{2(k-1)} f_j \otimes f_j - a_k I \right\| \leq C n^{-\epsilon_0}, \quad k \geq 1,$$

*for some  $a_k \in \mathbb{C}$ ,  $k \geq 1$  and  $C, \epsilon_0 > 0$  independent of  $n$  and  $N$ , then*

$$\mathbb{E} \circ \text{tr}(f_1 \otimes f_1 + \dots + f_N \otimes f_N)^p \rightarrow \sum_{\pi \in \text{NC}(p)} \prod_{B \in \pi} a_{|B|}.$$

Notation:  $\text{tr}$  means normalized trace.  $\text{NC}(p)$  is the set of all noncrossing partitions on  $\{1, \dots, p\}$ .

**Remarks.** 1. An immediate consequence of Theorem 1.1 is that the theorem of Marchenko and Pastur still holds if the random vector is distributed (but not uniformly distributed) on the unit sphere provided that it has bounded marginal  $L^4$  norms.

2. The condition  $\sup_{x \in S^{n-1}} \mathbb{E}|(f_i, x)|^4 \leq \frac{L}{n^2}$  cannot be removed from Theorem 1.1. For example, when  $N = n$  and each  $f_i$  is uniformly distributed on the canonical basis  $\{e_i\}_{i=1}^n$  for  $\mathbb{C}^n$ , we have  $a_k = 1$  and

$$\mathbb{E} \circ \text{tr}(f_1 \otimes f_1 + \dots + f_n \otimes f_n)^p \rightarrow B_p,$$

where  $B_p$  is the Bell number, the number of partitions on  $\{1, \dots, p\}$ .

## 2. A GRAPH INEQUALITY

This section is devoted to proving the following lemma.

**Lemma 2.1.** *Let  $S_1, \dots, S_r$  be subsets of a set  $E$  such that every element  $e \in E$  is contained in exactly two of the sets  $S_1, \dots, S_r$ . Assume that  $|S_1| \leq \dots \leq |S_r|$ . Let  $t \geq 0$ . Then*

$$\min(t, |S_1|) + \min(t, |S_2 \setminus S_1|) + \dots + \min(t, |S_r \setminus (S_1 \cup \dots \cup S_{r-1})|) \geq \frac{\min(t, |S_1|)}{2} r.$$

**Lemma 2.2.** *Let  $S_1, \dots, S_r$  be subsets of a set  $E$  such that every element  $x \in E$  is contained in exactly two of the sets  $S_1, \dots, S_r$ . Then*

$$|E| = \frac{1}{2} \sum_{k=1}^r |S_k|.$$

*Proof.* By assumption,  $\sum_{k=1}^r I_{S_k}(x) = 2$  for all  $x \in E$ . So

$$\sum_{k=1}^r |S_k| = \sum_{k=1}^r \sum_{x \in E} I_{S_k}(x) = \sum_{x \in E} \sum_{k=1}^r I_{S_k}(x) = \sum_{x \in E} 2 = 2|E|.$$

□

In Lemma 2.3 and 2.5 below,  $\Lambda^c$  is understood as  $\{1, \dots, r\} \setminus \Lambda$ . Also when  $k = 1$ ,  $S_k \setminus (S_1 \cup \dots \cup S_{k-1})$  is understood as  $S_1$ .

**Lemma 2.3.** *Let  $S_1, \dots, S_r$  be subsets of a set  $E$  such that every element  $x \in E$  is contained in exactly two of the sets  $S_1, \dots, S_r$ . If  $\Lambda \subset \{1, \dots, r\}$  and  $1 \leq k_0 \leq r$ , then*

$$\sum_{k \in \Lambda} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \geq \frac{1}{2} \left( \sum_{\substack{1 \leq k \leq k_0-1 \\ k \in \Lambda}} |S_k| - \sum_{\substack{1 \leq k \leq k_0-1 \\ k \in \Lambda^c}} |S_k| \right).$$

*Proof.*

$$\begin{aligned} & \sum_{k \in \Lambda} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \\ &= \sum_{k=1}^r |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| - \sum_{k \in \Lambda^c} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \\ &= |E| - \sum_{k \in \Lambda^c} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \text{ since } E = \bigcup_{k=1}^r S_k \\ &= \frac{1}{2} \sum_{k=1}^r |S_k| - \sum_{k \in \Lambda^c} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \text{ by Lemma 2.2} \\ &= \frac{1}{2} \sum_{k \in \Lambda} |S_k| + \frac{1}{2} \sum_{k \in \Lambda^c} |S_k| - \frac{1}{2} \sum_{k \in \Lambda^c} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| - \frac{1}{2} \sum_{k \in \Lambda^c} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \\ &= \frac{1}{2} \sum_{k \in \Lambda} |S_k| + \frac{1}{2} \sum_{k \in \Lambda^c} |S_k \cap (S_1 \cup \dots \cup S_{k-1})| - \frac{1}{2} \sum_{k \in \Lambda^c} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \\ &= \frac{1}{2} \sum_{k \in \Lambda} |S_k| + \frac{1}{2} \sum_{k \in \Lambda^c} |S_k \cap (S_1 \cup \dots \cup S_{k-1})| - \frac{1}{2} \sum_{\substack{1 \leq k \leq k_0-1 \\ k \in \Lambda^c}} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \sum_{\substack{k_0 \leq k \leq n \\ k \in \Lambda^c}} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \\
& \geq \frac{1}{2} \sum_{k \in \Lambda} |S_k| + \frac{1}{2} \sum_{k \in \Lambda^c} |S_k \cap (S_1 \cup \dots \cup S_{k-1})| - \frac{1}{2} \sum_{\substack{1 \leq k \leq k_0-1 \\ k \in \Lambda^c}} |S_k| \\
& \quad - \frac{1}{2} \sum_{k_0 \leq k \leq n} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \\
& = \frac{1}{2} \sum_{\substack{1 \leq k \leq k_0-1 \\ k \in \Lambda}} |S_k| + \frac{1}{2} \sum_{\substack{k_0 \leq k \leq n \\ k \in \Lambda}} |S_k| + \frac{1}{2} \sum_{k \in \Lambda^c} |S_k \cap (S_1 \cup \dots \cup S_{k-1})| - \frac{1}{2} \sum_{\substack{1 \leq k \leq k_0-1 \\ k \in \Lambda^c}} |S_k| \\
& \quad - \frac{1}{2} \sum_{k_0 \leq k \leq n} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \\
& = \frac{1}{2} \left( \sum_{\substack{1 \leq k \leq k_0-1 \\ k \in \Lambda}} |S_k| - \sum_{\substack{1 \leq k \leq k_0-1 \\ k \in \Lambda^c}} |S_k| \right) + \\
& \quad \frac{1}{2} \left( \sum_{\substack{k_0 \leq k \leq n \\ k \in \Lambda}} |S_k| + \sum_{k \in \Lambda^c} |S_k \cap (S_1 \cup \dots \cup S_{k-1})| - \sum_{k_0 \leq k \leq n} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \right).
\end{aligned}$$

To complete the proof, it suffices to show that

$$(2.1) \quad \sum_{\substack{k_0 \leq k \leq r \\ k \in \Lambda}} |S_k| + \sum_{k \in \Lambda^c} |S_k \cap (S_1 \cup \dots \cup S_{k-1})| - \sum_{k_0 \leq k \leq r} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \geq 0.$$

To begin,

$$\begin{aligned}
& \sum_{\substack{k_0 \leq k \leq r \\ k \in \Lambda}} |S_k| + \sum_{k \in \Lambda^c} |S_k \cap (S_1 \cup \dots \cup S_{k-1})| - \sum_{k_0 \leq k \leq r} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \\
& \geq \sum_{\substack{k_0 \leq k \leq r \\ k \in \Lambda}} |S_k \cap (S_1 \cup \dots \cup S_{k-1})| + \sum_{\substack{k_0 \leq k \leq r \\ k \in \Lambda^c}} |S_k \cap (S_1 \cup \dots \cup S_{k-1})| \\
& \quad - \sum_{k_0 \leq k \leq r} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \\
& = \sum_{k_0 \leq k \leq r} |S_k \cap (S_1 \cup \dots \cup S_{k-1})| - \sum_{k_0 \leq k \leq r} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \\
(2.2) \quad & = \sum_{k_0 \leq j \leq r} |S_j \cap (S_1 \cup \dots \cup S_{j-1})| - \sum_{k_0 \leq k \leq r} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})|.
\end{aligned}$$

By assumption, every element in  $V$  is contained in at least two of the sets  $S_1, \dots, S_r$ . Therefore, if an element  $e$  of  $S_k$  is not in  $S_1 \cup \dots \cup S_{k-1}$  then  $e$  must be in  $S_{k+1} \cup \dots \cup S_r$ . Thus,

$$|S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \leq |S_k \cap (S_{k+1} \cup \dots \cup S_r)| \leq \sum_{k+1 \leq j \leq r} |S_k \cap S_j|.$$

Hence,

$$\begin{aligned}
 \sum_{k_0 \leq k \leq r} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| &\leq \sum_{k_0 \leq k \leq r} \sum_{k+1 \leq j \leq n} |S_k \cap S_j| \\
 &= \sum_{k_0+1 \leq j \leq r} \sum_{k_0 \leq k \leq j-1} |S_k \cap S_j| \\
 (2.3) \qquad &\leq \sum_{k_0 \leq j \leq r} \sum_{1 \leq k \leq j-1} |S_k \cap S_j|.
 \end{aligned}$$

By assumption, every element in  $E$  is contained in at most two of the sets  $S_1, \dots, S_n$ . So the sets  $S_1 \cap S_j, \dots, S_{j-1} \cap S_j$  are disjoint. So  $\sum_{1 \leq k \leq j-1} |S_k \cap S_j| = |S_j \cap (S_1 \cup \dots \cup S_{j-1})|$ . Thus,

by (2.3),

$$\sum_{k_0 \leq k \leq r} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \leq \sum_{k_0 \leq j \leq r} |S_j \cap (S_1 \cup \dots \cup S_{j-1})|.$$

Combining this with (2.2), we obtain (2.1). This completes the proof.  $\square$

**Lemma 2.4.** *Let  $m \geq 1$ . Let  $\Lambda_1$  and  $\Lambda_2$  be subsets of  $\{1, \dots, m\}$ . If  $|[l, m] \cap \Lambda_1| \leq |[l, m] \cap \Lambda_2|$  for all  $l \in \{1, \dots, m\}$  then there exists a strictly increasing function  $f : \Lambda_1 \rightarrow \Lambda_2$  such that  $f(k) \geq k$  for all  $k \in \Lambda_1$ .*

*Proof.* Since by assumption  $|\Lambda_1| \leq |\Lambda_2|$ , the function  $f : \Lambda_1 \rightarrow \Lambda_2$  defined by sending the  $i$ th largest element of  $\Lambda_1$  to the  $i$ th largest element of  $\Lambda_2$  is well defined and strictly increasing. It remains to show that  $f(k) \geq k$  for all  $k \in \Lambda_1$ . For each  $i = 1, \dots, |\Lambda_1|$ , let  $k_i$  be the  $i$ th largest element of  $\Lambda_1$ . By assumption,  $|[k_i, m] \cap \Lambda_1| \leq |[k_i, m] \cap \Lambda_2|$  for all  $i = 1, \dots, |\Lambda_1|$ . Note that  $[k_i, m] \cap \Lambda_1 = \{k_1, k_2, \dots, k_i\}$ . So  $|[k_i, m] \cap \Lambda_1| = i$ . Therefore,  $|[k_i, m] \cap \Lambda_2| \geq i$  for all  $i = 1, \dots, |\Lambda_1|$ . So the  $i$ th largest element of  $\Lambda_2$  is at least  $k_i$ . So  $f(k_i) \geq k_i$  for all  $i = 1, \dots, |\Lambda_1|$  so  $f(k) \geq k$  for all  $k \in \Lambda_1$ .  $\square$

**Lemma 2.5.** *Let  $S_1, \dots, S_r$  be subsets of a set  $E$  such that every element  $x \in E$  is contained in exactly two of the sets  $S_1, \dots, S_r$ . Assume that  $|S_1| \leq \dots \leq |S_r|$ . If  $\Lambda \subset \{1, \dots, r\}$  then*

$$\sum_{k \in \Lambda} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \geq \frac{1}{2} |S_1| (|\Lambda| - |\Lambda^c|).$$

*Proof.* Case I: For every  $1 \leq l \leq r$ ,  $|[l, r] \cap \Lambda^c| < |[l, r] \cap \Lambda|$ .

From the first four lines of the proof of Lemma 2.3, we have

$$\sum_{k \in \Lambda} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| = \frac{1}{2} \sum_{k=1}^r |S_k| - \sum_{k \in \Lambda^c} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})|.$$

Thus,

$$\sum_{k \in \Lambda} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \geq \frac{1}{2} \sum_{k=1}^r |S_k| - \sum_{k \in \Lambda^c} |S_k| = \frac{1}{2} \sum_{k \in \Lambda} |S_k| - \frac{1}{2} \sum_{k \in \Lambda^c} |S_k|.$$

Taking  $m = r$ ,  $\Lambda_1 = \Lambda^c$  and  $\Lambda_2 = \Lambda$  in Lemma 2.4, we obtain an injective function  $f : \Lambda^c \rightarrow \Lambda$  such that  $f(k) \geq k$  for all  $k \in \Lambda^c$ . Therefore,

$$\begin{aligned}
 \sum_{k \in \Lambda} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| &= \frac{1}{2} \sum_{j \in \Lambda} |S_j| - \frac{1}{2} \sum_{k \in \Lambda^c} |S_k| \\
 &= \frac{1}{2} \sum_{j \in f(\Lambda^c)} |S_j| + \frac{1}{2} \sum_{j \in \Lambda \setminus f(\Lambda^c)} |S_j| - \frac{1}{2} \sum_{k \in \Lambda^c} |S_k|
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{k \in \Lambda^c} |S_{f(k)}| + \frac{1}{2} \sum_{j \in \Lambda \setminus f(\Lambda^c)} |S_j| - \frac{1}{2} \sum_{k \in \Lambda^c} |S_k| \\
&= \frac{1}{2} \sum_{k \in \Lambda^c} (|S_{f(k)}| - |S_k|) + \frac{1}{2} \sum_{j \in \Lambda \setminus f(\Lambda^c)} |S_j| \\
&\geq 0 + \frac{1}{2} |\Lambda \setminus f(\Lambda^c)| |S_1|.
\end{aligned}$$

The last inequality follows from the fact that  $f(k) \geq k$  for all  $k \in \Lambda^c$  and the assumption that  $|S_1| \leq \dots \leq |S_r|$ . Since  $|\Lambda \setminus f(\Lambda^c)| = |\Lambda| - |f(\Lambda^c)| = |\Lambda| - |\Lambda^c|$ , it follows that

$$\sum_{k \in \Lambda} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \geq \frac{1}{2} (|\Lambda| - |\Lambda^c|) |S_1|.$$

Case II: *There exists  $1 \leq k_0 \leq r$  such that  $[[k_0, r] \cap \Lambda^c] \geq [[k_0, r] \cap \Lambda]$ .*

We may assume that  $k_0$  is the smallest one with such property. We may also assume that  $k_0 > 1$ . Otherwise, the result is trivial. Thus, we have  $[[l, k_0 - 1] \cap \Lambda^c] < [[l, k_0 - 1] \cap \Lambda]$  for all  $l \in \{1, \dots, k_0 - 1\}$ . Otherwise, an  $l$  failing this property would contradict with the minimality of  $k_0$ . Taking  $m = k_0 - 1$ ,  $\Lambda_1 = [1, k_0 - 1] \cap \Lambda^c$  and  $\Lambda_2 = [1, k_0 - 1] \cap \Lambda$  in Lemma 2.4, we obtain an injective function  $f : [1, k_0 - 1] \cap \Lambda^c \rightarrow [1, k_0 - 1] \cap \Lambda$  satisfying  $f(k) \geq k$  for all  $k \in [1, k_0 - 1] \cap \Lambda^c$ .

By Lemma 2.3, we have

$$\begin{aligned}
&\sum_{k \in \Lambda} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \\
&\geq \frac{1}{2} \left( \sum_{\substack{1 \leq k \leq k_0 - 1 \\ k \in \Lambda^c}} |S_k| - \sum_{\substack{1 \leq k \leq k_0 - 1 \\ k \in \Lambda^c}} |S_k| \right) \\
&= \frac{1}{2} \left( \sum_{j \in [1, k_0 - 1] \cap \Lambda} |S_j| - \sum_{k \in [1, k_0 - 1] \cap \Lambda^c} |S_k| \right) \\
&= \frac{1}{2} \left( \sum_{j \in \{f(k) : k \in [1, k_0 - 1] \cap \Lambda^c\}} |S_j| + \sum_{j \in [1, k_0 - 1] \cap \Lambda \setminus \{f(k) : k \in [1, k_0 - 1] \cap \Lambda^c\}} |S_j| - \sum_{k \in [1, k_0 - 1] \cap \Lambda^c} |S_k| \right) \\
&= \frac{1}{2} \left( \sum_{k \in [1, k_0 - 1] \cap \Lambda^c} |S_{f(k)}| + \sum_{j \in [1, k_0 - 1] \cap \Lambda \setminus \{f(k) : k \in [1, k_0 - 1] \cap \Lambda^c\}} |S_j| - \sum_{k \in [1, k_0 - 1] \cap \Lambda^c} |S_k| \right) \\
&= \frac{1}{2} \left( \sum_{k \in [1, k_0 - 1] \cap \Lambda^c} (|S_{f(k)}| - |S_k|) + \sum_{j \in [1, k_0 - 1] \cap \Lambda \setminus \{f(k) : k \in [1, k_0 - 1] \cap \Lambda^c\}} |S_j| \right) \\
&\geq \frac{1}{2} (0 + |[1, k_0 - 1] \cap \Lambda \setminus \{f(k) : k \in [1, k_0 - 1] \cap \Lambda^c\}| |S_1|).
\end{aligned}$$

The last equality follows from the fact that  $f(k) \geq k$  for all  $k \in [1, k_0 - 1] \cap \Lambda^c$  and the assumption that  $|S_1| \leq \dots \leq |S_r|$ . Therefore,

$$\begin{aligned}
\sum_{k \in \Lambda} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| &\geq \frac{1}{2} (|[1, k_0 - 1] \cap \Lambda| - |\{f(k) : k \in [1, k_0 - 1] \cap \Lambda^c\}|) |S_1| \\
&= \frac{1}{2} (|[1, k_0 - 1] \cap \Lambda| - |[1, k_0 - 1] \cap \Lambda^c|) |S_1|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}(|\Lambda| - |[k_0, r] \cap \Lambda| - |\Lambda^c| + |[k_0, r] \cap \Lambda^c|)|S_1| \\
&\geq \frac{1}{2}(|\Lambda| - |\Lambda^c|)|S_1|.
\end{aligned}$$

The last inequality follows from Case II assumption.  $\square$

*Proof of Lemma 2.1.* Let  $\Lambda = \{1 \leq k \leq r : |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| \leq t\}$ . Then

$$\begin{aligned}
&\min(t, |S_1|) + \min(t, |S_2 \setminus S_1|) + \dots + \min(t, |S_n \setminus (S_1 \cup \dots \cup S_{n-1})|) \\
&= \sum_{k \in \Lambda} |S_k \setminus (S_1 \cup \dots \cup S_{k-1})| + t|\Lambda^c|.
\end{aligned}$$

If  $|\Lambda^c| \geq \frac{r}{2}$  then

$$\min(t, |S_1|) + \min(t, |S_2 \setminus S_1|) + \dots + \min(t, |S_r \setminus (S_1 \cup \dots \cup S_{r-1})|) \geq \frac{tr}{2}$$

and the result follows. If  $|\Lambda| \geq \frac{r}{2}$  then  $|\Lambda| - |\Lambda^c| \geq 0$  so by Lemma 2.5, it follows that

$$\begin{aligned}
&\min(t, |S_1|) + \min(t, |S_2 \setminus S_1|) + \dots + \min(t, |S_n \setminus (S_1 \cup \dots \cup S_{n-1})|) \\
&\geq \frac{1}{2}|S_1|(|\Lambda| - |\Lambda^c|) + t|\Lambda^c| \\
&\geq \frac{1}{2}\min(t, |S_1|)(|\Lambda| - |\Lambda^c|) + \min(t, |S_1|)|\Lambda^c| = \frac{1}{2}\min(t, |S_1|)(|\Lambda| + |\Lambda^c|) = \frac{\min(t, |S_1|)}{2}r.
\end{aligned}$$

$\square$

### 3. PROOF OF THE MAIN RESULT

**Lemma 3.1.** *If  $y$  and  $z$  are nonnegative random variables then for every  $0 < \epsilon < 1$ ,*

$$\mathbb{E}yz \leq (\mathbb{E}y)^{1-\epsilon}(\mathbb{E}y(z^{\frac{1}{\epsilon}}))^{\epsilon}.$$

*Proof.* By Hölder's inequality,  $\mathbb{E}yz = \mathbb{E}y^{1-\epsilon}(y^{\epsilon}z) \leq (\mathbb{E}y)^{1-\epsilon}(\mathbb{E}(y^{\epsilon}z)^{\frac{1}{\epsilon}})^{\epsilon} = (\mathbb{E}y)^{1-\epsilon}(\mathbb{E}y(z^{\frac{1}{\epsilon}}))^{\epsilon}$ .  $\square$

**Lemma 3.2.** *Let  $f_1, \dots, f_r$  be a random vector on  $\mathbb{C}^n$  such that for every  $\delta > 0$  there exists  $M_{\delta} > 0$  such that*

$$\sup_{x \in S^{n-1}} \mathbb{E}|(f, x)|^4 \leq \frac{M_{\delta}}{n^{2(1-\delta)}} \text{ and } \mathbb{E}\|f\|^k \leq L_k, \quad f \in \{f_1, \dots, f_r\}, \quad k \geq 1.$$

*Then for every  $\epsilon > 0$  and  $x_1, \dots, x_r \in \mathbb{C}^n$  with  $\|x_i\| \leq 1$ ,*

$$\mathbb{E}|(f_1, x_1)| \dots |(f_r, x_r)| \leq \frac{C_{\epsilon}}{n^{\frac{1}{2} \min(r, 4)(1-\epsilon)}},$$

*where  $C_{\epsilon}$  depends on  $\epsilon$  and certain  $M_{\delta}$  and  $L_{k, \delta}$  but not on  $n$ .*

*Proof.* By Hölder's inequality,

$$\mathbb{E}|(f_1, x_1)| \dots |(f_r, x_r)| \leq (\mathbb{E}|(f_1, x_1)|^r)^{\frac{1}{r}} \dots (\mathbb{E}|(f_r, x_r)|^r)^{\frac{1}{r}}$$

so it suffices to prove the lemma when  $f_1 = \dots = f_r = f$  and  $x_1 = \dots = x_r = x$ . If  $r > 4$  then by Lemma 3.1, for every  $\epsilon > 0$ ,

$$\begin{aligned}
\mathbb{E}|(f, x)|^r &\leq \mathbb{E}|(f, x)|^4 \|f\|^{r-4} \\
&\leq (\mathbb{E}|(f, x)|^4)^{1-\frac{\epsilon}{2}} (\mathbb{E}|(f, x)|^4 \|f\|^{\frac{2(r-4)}{\epsilon}})^{\frac{\epsilon}{2}} \\
&\leq (\mathbb{E}|(f, x)|^4)^{1-\frac{\epsilon}{2}} (\mathbb{E}\|f\|^{4+\frac{2(r-4)}{\epsilon}})^{\frac{\epsilon}{2}}
\end{aligned}$$

$$\begin{aligned}
 &\leq \left( \frac{M_{\frac{\epsilon}{2}}}{n^{2(1-\frac{\epsilon}{2})}} \right)^{1-\frac{\epsilon}{2}} (L_{4+\frac{2(r-4)}{\epsilon}})^{\frac{\epsilon}{2}} \\
 &\leq \frac{M_{\frac{\epsilon}{2}}^{1-\frac{\epsilon}{2}}}{n^{2(1-\epsilon)}} (L_{4+\frac{2(r-4)}{\epsilon}})^{\frac{\epsilon}{2}}.
 \end{aligned}$$

If  $r \leq 4$  then by Hölder's inequality,

$$\mathbb{E}|(f, x)|^r \leq (\mathbb{E}|(f, x)|^4)^{\frac{r}{4}} \leq \frac{M_{\frac{\epsilon}{2}}^{\frac{r}{4}}}{n^{\frac{r}{2}(1-\epsilon)}}.$$

□

**Lemma 3.3.** *Let  $G = (V, E)$  be a graph with no loops but perhaps with multiple edges. Let  $(\mathcal{B}_v)_{v \in V}$  be independent  $\sigma$ -subalgebras of a probability space  $(\Omega, \mathcal{B}, \mathbb{P})$ . For each  $e \in E$ , let  $u_1(e)$  and  $u_2(e)$  be the two endpoints of  $e$  and let  $h_e^{(1)}$  and  $h_e^{(2)}$  be  $\mathcal{B}_{u_1(e)}$ -measurable and  $\mathcal{B}_{u_2(e)}$ -measurable random vectors on  $\mathbb{C}^n$ . Assume that for every  $\delta > 0$ , there exist  $M_\delta > 0$  and  $L_{k,\delta}$  such that*

$$\sup_{x \in S^{n-1}} \mathbb{E}|(h, x)|^4 \leq \frac{M_\delta}{n^{2(1-\delta)}} \text{ and } \mathbb{E}\|h\|^k \leq L_{k,\delta} n^\delta, \quad h \in \bigcup_{e \in E} \{h_e^{(1)}, h_e^{(2)}\}, \quad k \geq 1.$$

If every vertex has degree at least 4, then for every  $\epsilon > 0$ ,

$$\mathbb{E} \prod_{e \in E} |\langle h_e^{(1)}, h_e^{(2)} \rangle| \leq \frac{C_\epsilon}{n^{|V|(1-\epsilon)}},$$

where  $C_\epsilon$  depends on  $\epsilon$ , the graph  $G$  and certain  $M_\delta$  and  $L_{k,\delta}$  but not on  $n$ .

*Proof.* Let  $v_1, \dots, v_{|V|}$  be an enumeration of  $V$  with ascending order according to their degrees, i.e., defining  $S_j$  to be the set of all edges incident to  $v_j$ , we have  $|S_1| \leq |S_2| \leq \dots |S_{|V|}|$ . For each  $j = 1, \dots, |V|$ , if  $e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})$  then either  $u_1(e) = v_j$  or  $u_2(e) = v_j$  and so by interchanging the values of  $u_1(e)$  and  $u_2(e)$  (and accordingly also  $h_e^{(1)}$  and  $h_e^{(2)}$ ), if necessary, we may assume that  $u_1(e) = v_j$ . Thus, for every  $\eta > 0$

$$\begin{aligned}
 &\mathbb{E} \prod_{e \in E} |\langle h_e^{(1)}, h_e^{(2)} \rangle| \\
 &= \mathbb{E} \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} |\langle h_e^{(1)}, h_e^{(2)} \rangle| \\
 &= \mathbb{E} \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| (\|h_e^{(2)}\| + \eta) \\
 &= \mathbb{E} \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} (\|h_e^{(2)}\| + \eta) \\
 (3.1) \quad &= \mathbb{E} \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \prod_{e \in E} (\|h_e^{(2)}\| + \eta),
 \end{aligned}$$

where as before, when  $j = 1$ ,  $S_j \setminus (S_1 \cup \dots \cup S_{j-1})$  is understood as  $S_1$ .

Since  $u_1(e) = v_j$ ,  $h_e^{(1)}$  is  $\mathcal{B}_{v_j}$ -measurable. On the other hand, by assumption,  $h_e^{(2)}$  is  $\mathcal{B}_{u_2(e)}$ -measurable; and since  $G$  has no loops,  $u_2(e) \neq u_1(e) = v_j$ . Therefore, by Lemma 3.2,

$$(3.2) \quad \mathbb{E}_{\mathcal{B}_{v_j}} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \leq \frac{C_\epsilon}{n^{\frac{1}{2} \min(|S_j \setminus (S_1 \cup \dots \cup S_{j-1})|, 4)(1-\epsilon)}}.$$

Note that the right hand side is a constant. We claim that

$$(3.3) \quad \mathbb{E} \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \leq \frac{C_\epsilon}{n^{|V|(1-\epsilon)}},$$

where  $C_\epsilon$  denotes any positive number depending on  $\epsilon$ , the graph  $G$  and certain  $M_\delta$  and  $L_{k,\delta}$  but not on  $n$ .

To prove the claim, we write

$$\begin{aligned} & \mathbb{E} \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \\ &= \mathbb{E} \left( \prod_{e \in S_1} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \right) \left( \prod_{j=2}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \right). \end{aligned}$$

All the edges  $e$  in the first parenthesis are incident to  $v_1$ , whereas all the  $e$  in the second parenthesis are not incident to  $v_1$ . Thus, the term in the second parenthesis is independent of  $\mathcal{B}_{v_1}$  and so

$$\begin{aligned} & \mathbb{E} \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \\ &= \mathbb{E} \left( \mathbb{E}_{\mathcal{B}_{v_1}} \prod_{e \in S_1} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \right) \left( \prod_{j=2}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \right) \\ &\leq \frac{C_\epsilon}{n^{\frac{1}{2} \min(|S_1|, 4)(1-\epsilon)}} \mathbb{E} \left( \prod_{j=2}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \right), \end{aligned}$$

where the inequality follows from (3.2). Continuing this procedure, we obtain

$$\begin{aligned} & \mathbb{E} \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \\ &\leq \frac{C_\epsilon}{n^{\frac{1}{2} \min(|S_1|, 4)(1-\epsilon)}} \frac{C_\epsilon}{n^{\frac{1}{2} \min(|S_2 \setminus S_1|, 4)(1-\epsilon)}} \cdots \frac{C_\epsilon}{n^{\frac{1}{2} \min(|S_{|V|} \setminus (S_1 \cup \dots \cup S_{|V|-1})|, 4)(1-\epsilon)}}. \end{aligned}$$

By Lemma 2.1, it follows that

$$\mathbb{E} \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \leq \frac{C_\epsilon}{n^{\frac{1}{4} \min(|S_1|, 4)|V|(1-\epsilon)}},$$



possibly with different  $C_\epsilon$ . Since by assumption,  $|S_1| \geq 4$ , the claim (3.3) is proved. Having proved (3.3), before we apply Lemma 3.1, we estimate

$$\begin{aligned}
 & \mathbb{E} \left( \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \left| \left\langle h_e^{(1)}, \frac{h_e^{(2)}}{\|h_e^{(2)}\| + \eta} \right\rangle \right| \right) \left( \prod_{e \in E} (\|h_e^{(2)}\| + \eta) \right)^{\frac{1}{\epsilon}} \\
 & \leq \mathbb{E} \left( \prod_{j=1}^{|V|} \prod_{e \in S_j \setminus (S_1 \cup \dots \cup S_{j-1})} \|h_e^{(1)}\| \right) \left( \prod_{e \in E} (\|h_e^{(2)}\| + \eta)^{\frac{1}{\epsilon}} \right) \\
 & \leq \mathbb{E} \left( \prod_{e \in E} \|h_e^{(1)}\| \right) \left( \prod_{e \in E} (\|h_e^{(2)}\| + \eta)^{\frac{1}{\epsilon}} \right) \\
 & \leq \left( \prod_{e \in E} \mathbb{E} \|h_e^{(1)}\|^{2|E|} \prod_{e \in E} \mathbb{E} (\|h_e^{(2)}\| + \eta)^{\frac{2|E|}{\epsilon}} \right)^{\frac{1}{2|E|}},
 \end{aligned}$$

where the last inequality follows from Hölder's inequality. Combining this estimate with (3.1), (3.3) and Lemma 3.1, we obtain

$$\mathbb{E} \prod_{e \in E} |\langle h_e^{(1)}, h_e^{(2)} \rangle| \leq \frac{C_\epsilon}{n^{|V|(1-\epsilon)^2}} \left( \prod_{e \in E} \mathbb{E} \|h_e^{(1)}\|^{2|E|} \prod_{e \in E} \mathbb{E} (\|h_e^{(2)}\| + \eta)^{\frac{2|E|}{\epsilon}} \right)^{\frac{\epsilon}{2|E|}}.$$

Taking  $\eta$  to be arbitrarily small, we have

$$\begin{aligned}
 \mathbb{E} \prod_{e \in E} |\langle h_e^{(1)}, h_e^{(2)} \rangle| & \leq \frac{C_\epsilon}{n^{|V|(1-\epsilon)^2}} \left( \prod_{e \in E} \mathbb{E} \|h_e^{(1)}\|^{2|E|} \prod_{e \in E} \mathbb{E} \|h_e^{(2)}\|^{\frac{2|E|}{\epsilon}} \right)^{\frac{\epsilon}{2|E|}} \\
 & \leq \frac{C_\epsilon}{n^{|V|(1-\epsilon)^2}} \left( \prod_{e \in E} (L_{2|E|,1} n) \prod_{e \in E} (L_{\frac{2|E|}{\epsilon},1} n) \right)^{\frac{\epsilon}{2|E|}} \\
 & \leq \frac{C_\epsilon}{n^{|V|(1-\epsilon)^2}} \left( \prod_{e \in E} L_{2|E|,1} \prod_{e \in E} L_{\frac{2|E|}{\epsilon},1} \right)^{\frac{\epsilon}{2|E|}} n^\epsilon,
 \end{aligned}$$

where the second inequality follows from the assumption. This completes the proof with a different  $\epsilon$ .  $\square$

**Lemma 3.4.** *Suppose that  $(\mathcal{B}_j)_{j \in J}$  are independent  $\sigma$ -subalgebras of a probability space  $(\Omega, \mathcal{B}, \mathbb{P})$ . Let  $j : \{1, \dots, p\} \rightarrow J$  be such that  $\ker j$  is a crossing partition on  $\{1, \dots, p\}$ . For each  $i = 1, \dots, p$ , let  $f_i^{(1)}, f_i^{(2)}$  be  $\mathcal{B}_{j(i)}$ -measurable functions on  $\Omega$ . Assume that for every  $\delta > 0$ , there exist  $M_\delta > 0$  and  $L_{k,\delta} > 0$ ,  $k \geq 1$  such that*

$$\sup_{x \in S^{n-1}} \mathbb{E} |(f, x)|^4 \leq \frac{M_\delta}{n^{2(1-\delta)}} \text{ and } \mathbb{E} \|f\|^k \leq L_{k,\delta} n^\delta, \quad f \in \{f_1^{(1)}, f_1^{(2)}, \dots, f_p^{(1)}, f_p^{(2)}\}, \quad k \geq 1$$

Then for every  $\epsilon > 0$ ,

$$|\mathbb{E} \circ \text{tr}(f_1^{(1)} \otimes f_1^{(2)})(f_2^{(1)} \otimes f_2^{(2)}) \dots (f_p^{(1)} \otimes f_p^{(2)})| \leq \frac{C_\epsilon}{n^{|j(1), \dots, j(p)|+1-\epsilon}},$$

where  $C_\epsilon > 0$  depends on  $\epsilon, p$  and certain  $M_\delta$  and  $L_{k,\delta}$  but not on  $n$ .

*Proof.* We may assume that  $j(1) \neq j(2) \neq \dots \neq j(p) \neq j(1)$  and each  $j(i)$  appears at least twice in the list  $j(1), \dots, j(p)$ . Otherwise, if  $j(i) = j(i+1)$  then

$$(f_i^{(1)} \otimes f_i^{(2)})(f_{i+1}^{(1)} \otimes f_{i+1}^{(2)}) = \langle f_{i+1}^{(1)}, f_i^{(2)} \rangle (f_i^{(1)} \otimes f_{i+1}^{(2)}) = (\langle f_{i+1}^{(1)}, f_i^{(2)} \rangle f_i^{(1)}) \otimes f_{i+1}^{(2)}.$$

Note that  $\langle f_{i+1}^{(1)}, f_i^{(2)} \rangle f_i^{(1)}$  and  $f_{i+1}^{(2)}$  are  $\mathcal{B}_{j(i)}$ -measurable since  $j(i) = j(i+1)$ . Also, by Hölder's inequality,  $\langle f_{i+1}^{(1)}, f_i^{(2)} \rangle f_i^{(1)}$  satisfies (3.4) perhaps with different  $M_\delta$  and  $L_{k,\delta}$ . Thus, the result follows by induction hypothesis since the product  $(f_1^{(1)} \otimes f_1^{(2)}) \dots (f_{p-1}^{(1)} \otimes f_p^{(2)})$  of  $p$  terms becomes a product of  $p-1$  terms. (The  $i$ th term and the  $(i+1)$ th term are combined.)

Similar argument works if we have  $j(p) \neq j(1)$ .

If there is a  $j(i)$  that appears only once in the list  $j(1), \dots, j(p)$ , then by independence of  $(\mathcal{B}_j)_{j \in J}$ ,

$$\begin{aligned}
 & \mathbb{E} \circ \text{tr}(f_1^{(1)} \otimes f_1^{(2)}) \dots (f_{p-1}^{(1)} \otimes f_p^{(2)}) \\
 &= \mathbb{E} \circ \text{tr}(f_1^{(1)} \otimes f_1^{(2)}) \dots (f_i^{(1)} \otimes f_i^{(2)}) \dots (f_{p-1}^{(1)} \otimes f_p^{(2)}) \\
 &= \mathbb{E} \circ \text{tr}(f_1^{(1)} \otimes f_1^{(2)}) \dots (\mathbb{E} f_i^{(1)} \otimes f_i^{(2)}) \dots (f_{p-1}^{(1)} \otimes f_p^{(2)}) \\
 &= \mathbb{E} \circ \text{tr}(f_1^{(1)} \otimes f_1^{(2)}) \dots (\mathbb{E} f_i^{(1)} \otimes f_i^{(2)}) (f_{i+1}^{(1)} \otimes f_{i+1}^{(2)}) \dots (f_{p-1}^{(1)} \otimes f_p^{(2)}) \\
 &= \mathbb{E} \circ \text{tr}(f_1^{(1)} \otimes f_1^{(2)}) \dots ((\mathbb{E} f_i^{(1)} \otimes f_i^{(2)}) f_{i+1}^{(1)} \otimes f_{i+1}^{(2)}) \dots (f_p^{(1)} \otimes f_p^{(2)}) \\
 (3.5) \quad &= \frac{1}{n} \mathbb{E} \circ \text{tr}(f_1^{(1)} \otimes f_1^{(2)}) \dots (n(\mathbb{E} f_i^{(1)} \otimes f_i^{(2)}) f_{i+1}^{(1)} \otimes f_{i+1}^{(2)}) \dots (f_p^{(1)} \otimes f_p^{(2)}).
 \end{aligned}$$

Note that  $\mathbb{E} f_i^{(1)} \otimes f_i^{(2)}$  is a deterministic matrix and

$$\begin{aligned}
 |\langle (\mathbb{E} f_i^{(1)} \otimes f_i^{(2)}) x, y \rangle| &= |\mathbb{E} \langle x, f_i^{(2)} \rangle \langle f_i^{(1)}, y \rangle| \\
 &\leq \mathbb{E} |\langle x, f_i^{(2)} \rangle| |\langle f_i^{(1)}, y \rangle| \\
 &\leq (\mathbb{E} |\langle f_i^{(2)}, x \rangle|^2)^{\frac{1}{2}} (\mathbb{E} |\langle f_i^{(1)}, y \rangle|^2)^{\frac{1}{2}} \\
 &\leq (\mathbb{E} |\langle f_i^{(2)}, x \rangle|^4)^{\frac{1}{4}} (\mathbb{E} |\langle f_i^{(1)}, y \rangle|^4)^{\frac{1}{4}} \\
 &\leq \left( \frac{M_\delta}{n^{2(1-\delta)}} \right)^{\frac{1}{4}} \left( \frac{M_\delta}{n^{2(1-\delta)}} \right)^{\frac{1}{4}} \\
 &= \frac{\sqrt{M_\delta}}{n^{1-\delta}}, \quad x, y \in S^{n-1}.
 \end{aligned}$$

Thus,

$$\|n \mathbb{E} f_i^{(1)} \otimes f_i^{(2)}\| \leq \sqrt{M_\delta} n^\delta.$$

Hence,  $n(\mathbb{E} f_i^{(1)} \otimes f_i^{(2)}) f_{i+1}^{(1)}$  is  $\mathcal{B}_{j(i+1)}$  and still satisfies (3.4) perhaps with different  $M_\delta$  and  $L_{k,\delta}$ . Thus, in view of (3.5), the result follows by induction hypothesis since the product  $(f_1^{(1)} \otimes f_1^{(2)}) \dots (f_p^{(1)} \otimes f_p^{(2)})$  of  $p$  terms becomes a product of  $p-1$  terms. (The  $i$ th term is absorbed by the  $(i+1)$ th term.)

Therefore, we may justifiably assume that  $j(1) \neq j(2) \neq \dots \neq j(p) \neq j(1)$  and each  $j(i)$  appears at least twice in the list  $j(1), \dots, j(p)$ .

$$\begin{aligned}
 & |\mathbb{E} \circ \text{tr}(f_1^{(1)} \otimes f_1^{(2)}) (f_2^{(1)} \otimes f_2^{(2)}) \dots (f_p^{(1)} \otimes f_p^{(2)})| \\
 &= \frac{1}{n} |\mathbb{E} \langle f_1^{(2)}, f_2^{(1)} \rangle \langle f_2^{(2)}, f_3^{(1)} \rangle \dots \langle f_p^{(2)}, f_1^{(1)} \rangle| \leq \frac{1}{n} \mathbb{E} |\langle f_1^{(2)}, f_2^{(1)} \rangle| |\langle f_2^{(2)}, f_3^{(1)} \rangle| \dots |\langle f_p^{(2)}, f_1^{(1)} \rangle|.
 \end{aligned}$$

For notational convenience, let  $j(p+1) = j(1)$  and  $f_{p+1}^{(1)} = f_1^{(1)}$ . Then we have

$$(3.6) \quad |\mathbb{E} \circ \text{tr}(f_1^{(1)} \otimes f_1^{(2)}) (f_2^{(1)} \otimes f_2^{(2)}) \dots (f_p^{(1)} \otimes f_p^{(2)})| \leq \frac{1}{n} \mathbb{E} \prod_{i=1}^p |\langle f_i^{(2)}, f_{i+1}^{(1)} \rangle|.$$

We use Lemma 3.3 to estimate this. First, we take the vertex set  $V = \{j(1), \dots, j(p)\}$  and the edge set  $E = \{1, \dots, p\}$ , where for each  $i \in E$ , the two endpoints are  $u_1(i) = j(i)$  and

$u_2(i) = j(i+1)$ . There are no loops since we assume that  $j(i) \neq j(i+1)$  for all  $i = 1, \dots, p$ . For each  $i \in E$ , take  $h_i^{(1)} = f_i^{(2)}$  and  $h_i^{(2)} = f_{i+1}^{(1)}$ . To see that every vertex has degree at least 4, recall that we assume that for every  $j \in V = \{j(1), \dots, j(p)\}$ , there exist  $i_1 \neq i_2$  in  $\{1, \dots, p\}$  such that  $j(i_1) = j(i_2) = j$ . Since  $j(1) \neq j(2) \neq \dots \neq j(p) \neq j(1)$ ,  $i_1$  and  $i_2$  cannot be consecutive numbers. Therefore, the vertex  $j$  is incident with the four distinct edges  $i_1 - 1, i_1, i_2 - 1, i_2$ . (When  $i_1 = 1$ ,  $i_1 - 1 = p$ .) Thus, the assumptions of Lemma 3.3 are satisfied and so we obtain

$$\mathbb{E} \prod_{i=1}^p |\langle f_i^{(2)}, f_{i+1}^{(1)} \rangle| \leq \frac{C_\epsilon}{n^{|\{j(1), \dots, j(p)\}|(1-\epsilon)}}.$$

The result follows by combining this with 3.6.  $\square$

**Remark.** In Lemma 3.4, the assumption that  $\ker j$  is a crossing partition is necessary because it guarantees that repeating the procedure of (1) combining the  $i$ th term and the  $(i+1)$ th term when  $j(i) = j(i+1)$  and (2) the  $i$ th term being absorbed by the  $(i+1)$ th term when  $j(i)$  appears only once in the list  $j(1), \dots, j(p)$  does not make reduce  $\{1, \dots, p\}$  to a singleton. Without the crossing assumption, one would have got Lemma 3.6 below.

As an immediate consequence of Lemma 3.4, we have

**Proposition 3.5.** *Suppose that  $(f_j)_{j \in J}$  is an independent family of random vectors on  $\mathbb{C}^n$  such that*

$$\sup_{x \in S^{n-1}} \mathbb{E}|(f_j, x)|^4 \leq \frac{L}{n^2} \text{ and } \mathbb{E}\|f_j\|^k \leq L_k, \quad j \in J, k \geq 1$$

for some  $L > 0$  and  $L_k > 0$ ,  $k \geq 1$  independent of  $N$ . Let  $j : \{1, \dots, p\} \rightarrow J$  be such that  $\ker j$  is a crossing partition on  $\{1, \dots, p\}$ . Then for every  $\epsilon > 0$ ,

$$|\mathbb{E} \circ \text{tr}(f_{j(1)} \otimes f_{j(1)}) \dots (f_{j(p)} \otimes f_{j(p)})| \leq \frac{C_\epsilon}{n^{|\{j(1), \dots, j(p)\}|+1-\epsilon}},$$

where  $C_\epsilon > 0$  depends on  $\epsilon, p, L$  and certain  $L_k$  but not on  $n$ .

The following lemma is the analog of Lemma 3.4 for noncrossing partition.

**Lemma 3.6.** *Suppose that  $(\mathcal{B}_j)_{j \in J}$  are independent  $\sigma$ -subalgebras of a probability space  $(\Omega, \mathcal{B}, \mathbb{P})$ . Let  $j : \{1, \dots, p\} \rightarrow J$  be such that  $\ker j$  is a noncrossing partition on  $\{1, \dots, p\}$ . For each  $i = 1, \dots, p$ , let  $f_i^{(1)}, f_i^{(2)}$  be  $\mathcal{B}_{j(i)}$ -measurable functions on  $\Omega$ . Assume that for every  $\delta > 0$ , there exist  $M_\delta > 0$  and  $L_{k,\delta} > 0$ ,  $k \geq 1$  such that*

$$\sup_{x \in S^{n-1}} \mathbb{E}|(f, x)|^4 \leq \frac{M_\delta}{n^{2(1-\delta)}} \text{ and } \mathbb{E}\|f\|^k \leq L_{k,\delta} n^\delta, \quad f \in \{f_1^{(1)}, f_1^{(2)}, \dots, f_p^{(1)}, f_p^{(2)}\}, k \geq 1.$$

Then for every  $\epsilon > 0$ ,

$$(3.7) \quad \|\mathbb{E}(f_1^{(1)} \otimes f_1^{(2)})(f_2^{(1)} \otimes f_2^{(2)}) \dots (f_p^{(1)} \otimes f_p^{(2)})\| \leq \frac{C_\epsilon}{n^{|\{j(1), \dots, j(p)\}|-\epsilon}},$$

where  $C_\epsilon > 0$  depends on  $\epsilon, p$  and certain  $M_\delta$  and  $L_{k,\delta}$  but not on  $n$ .

The only differences are that on the left hand side of (3.7), one has norm of expectation instead of trace expectation and that on the right hand side of (3.7), one only has  $\frac{C_\epsilon}{n^{|\{j(1), \dots, j(p)\}|-\epsilon}}$  instead of  $\frac{C_\epsilon}{n^{|\{j(1), \dots, j(p)\}|+1-\epsilon}}$  in Lemma 3.4. The proof of Lemma 3.6 is exactly the same as the beginning of the proof of Lemma 3.4. One needs the fact that for every noncrossing partition  $\pi$  on  $\{1, \dots, p\}$ , at least one of the following holds.

- (1) There exists  $i \in \{1, \dots, p-1\}$  such that  $i$  and  $i+1$  are in the same block of  $\pi$ .
- (2)  $\pi$  has a singleton block.

This is because every noncrossing partition contains an interval block.

As an immediate consequence of Lemma 3.7, we have

**Lemma 3.7.** *Suppose that  $(f_j)_{j \in J}$  is an independent family of random vectors on  $\mathbb{C}^n$  such that*

$$\sup_{x \in S^{n-1}} \mathbb{E}|(f_j, x)|^4 \leq \frac{L}{n^2} \text{ and } \mathbb{E}\|f_j\|^k \leq L_k, \quad j \in J, k \geq 1$$

*for some  $L > 0$  and  $L_k > 0$ ,  $k \geq 1$  independent of  $N$ . Let  $j : \{1, \dots, p\} \rightarrow J$  be such that  $\ker j$  is a noncrossing partition on  $\{1, \dots, p\}$ . Then for every  $\epsilon > 0$ ,*

$$\|\mathbb{E}(f_{j(1)} \otimes f_{j(1)}) \dots (f_{j(p)} \otimes f_{j(p)})\| \leq \frac{C_\epsilon}{n^{|\{j(1), \dots, j(p)\}| - \epsilon}},$$

*where  $C_\epsilon > 0$  depends on  $\epsilon, p, L$  and certain  $L_k$  but not on  $n$ .*

**Proposition 3.8.** *Suppose that  $f_1, \dots, f_N$  are independent random vectors on  $\mathbb{C}^n$  such that*

$$\sup_{x \in S^{n-1}} \mathbb{E}|(f_j, x)|^4 \leq \frac{L}{n^2} \text{ and } \mathbb{E}\|f_j\|^k \leq L_k, \quad j = 1, \dots, N, k \geq 1$$

*for some  $L > 0$  and  $L_k > 0$ ,  $k \geq 1$  independent of  $n$  and  $N$ . If  $n, N \rightarrow \infty$  in such a way that  $\frac{n}{N} \rightarrow \lambda \in (0, \infty)$  and*

$$n^{\epsilon_0} \left\| \sum_{j=1}^N \mathbb{E}\|f_j\|^{2(k-1)} f_j \otimes f_j - a_k I \right\| \rightarrow 0, \quad k \geq 1,$$

*for some  $a_k \in \mathbb{C}$ ,  $k \geq 1$  and  $\epsilon_0 > 0$  independent of  $n$  and  $N$ , then for every noncrossing partition  $\pi$  on  $\{1, \dots, p\}$ ,*

$$\left| \sum_{\substack{j: \{1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi}} \mathbb{E} \circ \text{tr}(f_{j(1)} \otimes f_{j(1)}) \dots (f_{j(p)} \otimes f_{j(p)}) - \prod_{B \in \pi} a_{|B|} \right| \rightarrow 0.$$

*Proof.* We prove by induction on  $p$ . For  $p = 1$ , the result is obvious. For  $p \geq 2$ , since  $\pi$  is a noncrossing partition on  $\{1, \dots, p\}$ , there is an interval block  $B_0 \in \pi$ . For simplicity, since the trace is cyclic invariant, we may assume that  $B_0 = \{1, \dots, q\}$  for some  $1 \leq q \leq p$ . Thus, for every  $j : \{1, \dots, p\} \rightarrow \{1, \dots, N\}$  with  $\ker j = \pi$ , we have

$$\begin{aligned} & \mathbb{E} \circ \text{tr}(f_{j(1)} \otimes f_{j(1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \\ &= \text{tr} \mathbb{E}(f_{j(1)} \otimes f_{j(1)}) \dots (f_{j(q)} \otimes f_{j(q)}) \mathbb{E}(f_{j(q+1)} \otimes f_{j(q+1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \\ &= \text{tr} \mathbb{E}(\|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)}) \mathbb{E}(f_{j(q+1)} \otimes f_{j(q+1)}) \dots (f_{j(p)} \otimes f_{j(p)}), \end{aligned}$$

since  $j(1) = \dots = j(q)$ . Note that every  $j : \{1, \dots, p\} \rightarrow \{1, \dots, N\}$  with  $\ker j = \pi$  corresponds to  $j : \{q+1, \dots, p\} \rightarrow \{1, \dots, N\}$  with  $\ker l = \pi \setminus \{B_0\}$  and  $j(1) \in \{1, \dots, N\} \setminus \{j(q+1), \dots, j(p)\}$ . Thus,

$$\begin{aligned} & \sum_{\substack{j: \{1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi}} \mathbb{E} \circ \text{tr}(f_{j(1)} \otimes f_{j(1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \\ &= \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \sum_{j(1) \in \{1, \dots, N\} \setminus \{j(q+1), \dots, j(p)\}} \\ & \quad \text{tr} \mathbb{E}(\|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)}) \mathbb{E}(f_{j(q+1)} \otimes f_{j(q+1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \sum_{j(1) \in \{1, \dots, N\}} \\
&\quad \text{tr} \mathbb{E}(\|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)}) \mathbb{E}(f_{j(q+1)} \otimes f_{j(q+1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \\
&\quad - \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \sum_{j(1) \in \{j(q+1), \dots, j(p)\}} \\
&\quad \text{tr} \mathbb{E}(\|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)}) \mathbb{E}(f_{j(q+1)} \otimes f_{j(q+1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \\
&= \text{tr} \left( \sum_{j(1) \in \{1, \dots, N\}} \mathbb{E} \|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)} \right) \\
&\quad \left( \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \mathbb{E}(f_{j(q+1)} \otimes f_{j(q+1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \right) \\
&\quad - \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \sum_{j(1) \in \{j(q+1), \dots, j(p)\}} \\
&\quad \text{tr} \mathbb{E}(\|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)}) \mathbb{E}(f_{j(q+1)} \otimes f_{j(q+1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \\
&= \text{tr} a_q I \left( \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \mathbb{E}(f_{j(q+1)} \otimes f_{j(q+1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \right) \\
&\quad + \text{tr} \left( \sum_{j(1) \in \{1, \dots, N\}} \mathbb{E} \|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)} - a_q I \right) \\
&\quad \left( \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \mathbb{E}(f_{j(q+1)} \otimes f_{j(q+1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \right) \\
&\quad - \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \sum_{j(1) \in \{j(q+1), \dots, j(p)\}} \\
&\quad \text{tr} \mathbb{E}(\|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)}) \mathbb{E}(f_{j(q+1)} \otimes f_{j(q+1)}) \dots (f_{j(p)} \otimes f_{j(p)}).
\end{aligned}$$

By induction hypothesis, the first term

$$\text{tr} a_q I \left( \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \mathbb{E}(f_{j(q+1)} \otimes f_{j(q+1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \right)$$

converges to  $a_q \prod_{B \in \pi \setminus \{B_0\}} a_{|B|} = \prod_{B \in \pi} a_{|B|}$ . For the second term,

$$\begin{aligned}
& \left| \operatorname{tr} \left( \sum_{j(1) \in \{1, \dots, N\}} \mathbb{E} \|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)} - a_q I \right) \right. \\
& \quad \left. \left( \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \mathbb{E} (f_{j(q+1)} \otimes f_{j(q+1)}) \cdots (f_{j(p)} \otimes f_{j(p)}) \right) \right| \\
& \leq \left\| \sum_{j(1) \in \{1, \dots, N\}} \mathbb{E} \|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)} - a_q I \right\| \\
& \quad \left( \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \|\mathbb{E} (f_{j(q+1)} \otimes f_{j(q+1)}) \cdots (f_{j(p)} \otimes f_{j(p)})\| \right) \\
& \leq \left\| \sum_{j(1) \in \{1, \dots, N\}} \mathbb{E} \|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)} - a_q I \right\| \\
& \quad \left( \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \frac{C_{\frac{\epsilon_0}{2}}}{n^{|\{j(q+1), \dots, j(p)\}| - \frac{\epsilon_0}{2}}} \right) \\
& \quad \text{by Lemma 3.7} \\
& \leq \left\| \sum_{j(1) \in \{1, \dots, N\}} \mathbb{E} \|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)} - a_q I \right\| C_{\frac{\epsilon_0}{2}} n^{\frac{\epsilon_0}{2}} \rightarrow 0.
\end{aligned}$$

For the third term,

$$\begin{aligned}
& \left| \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \sum_{j(1) \in \{j(q+1), \dots, j(p)\}} \right. \\
& \quad \left. \operatorname{tr} \mathbb{E} (\|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)}) \mathbb{E} (f_{j(q+1)} \otimes f_{j(q+1)}) \cdots (f_{j(p)} \otimes f_{j(p)}) \right| \\
& \leq \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \sum_{j(1) \in \{j(q+1), \dots, j(p)\}} \\
& \quad \|\mathbb{E} (\|f_{j(1)}\|^{2(q-1)} f_{j(1)} \otimes f_{j(1)})\| \|\mathbb{E} (f_{j(q+1)} \otimes f_{j(q+1)}) \cdots (f_{j(p)} \otimes f_{j(p)})\| \\
& \quad \sum_{\substack{j: \{q+1, \dots, p\} \rightarrow \{1, \dots, N\} \\ \ker j = \pi \setminus \{B\}}} \sum_{j(1) \in \{j(q+1), \dots, j(p)\}} \frac{C_{\frac{1}{4}}}{n^{1-\frac{1}{4}}} \frac{C_{\frac{1}{4}}}{n^{|\{j(q+1), \dots, j(p)\}| - \frac{1}{4}}} \\
& \quad \text{by Lemma 3.7 with } \epsilon = \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{\substack{j:\{q+1,\dots,p\}\rightarrow\{1,\dots,N\} \\ \ker j=\pi\setminus\{B\}}} p^{\frac{C_1}{4}} \frac{C_1}{n^{1-\frac{1}{4}} n^{|\{j(q+1),\dots,j(p)\}|-\frac{1}{4}}} \\
 &= \sum_{\substack{j:\{q+1,\dots,p\}\rightarrow\{1,\dots,N\} \\ \ker j=\pi\setminus\{B\}}} \frac{C}{n^{|\{j(q+1),\dots,j(p)\}|+\frac{1}{2}}} \leq \frac{C}{n^{\frac{1}{2}}} \rightarrow 0.
 \end{aligned}$$

□

*Proof of Theorem 1.1.*

$$\begin{aligned}
 \mathbb{E} \circ \text{tr}(f_1 \otimes f_1 + \dots + f_N \otimes f_N)^p &= \sum_{j:\{1,\dots,p\}\rightarrow\{1,\dots,N\}} \mathbb{E} \circ \text{tr}(f_{j(1)} \otimes f_{j(1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \\
 &= \sum_{\substack{j:\{1,\dots,p\}\rightarrow\{1,\dots,N\} \\ \ker j \text{ noncrossing}}} \mathbb{E} \circ \text{tr}(f_{j(1)} \otimes f_{j(1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \\
 &\quad + \sum_{\substack{j:\{1,\dots,p\}\rightarrow\{1,\dots,N\} \\ \ker j \text{ crossing}}} \mathbb{E} \circ \text{tr}(f_{j(1)} \otimes f_{j(1)}) \dots (f_{j(p)} \otimes f_{j(p)})
 \end{aligned}$$

The first term converges to  $\sum_{\pi \in \text{NC}(p)} \prod_{B \in \pi} a_{|B|}$  by Proposition 3.8. For the second term,

$$\begin{aligned}
 &\sum_{\substack{j:\{1,\dots,p\}\rightarrow\{1,\dots,N\} \\ \ker j \text{ crossing}}} \mathbb{E} \circ \text{tr}(f_{j(1)} \otimes f_{j(1)}) \dots (f_{j(p)} \otimes f_{j(p)}) \\
 &\leq \sum_{\substack{j:\{1,\dots,p\}\rightarrow\{1,\dots,N\} \\ \ker j \text{ crossing}}} \frac{C_1}{n^{|\{j(1),\dots,j(p)\}|+1-\frac{1}{2}}} \text{ by Proposition 3.5 with } \epsilon = \frac{1}{2} \\
 &\leq \frac{C}{n^{\frac{1}{2}}} \rightarrow 0.
 \end{aligned}$$

□

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